J. Fluid Mech. (1986), *vol.* 169, *pp.* 251–256
Printed in Great Britain

Note on added mass and drift

By T. BROOKE BENJAMIN
Mathematical Institute, 24/29 St Giles, Oxford OX1 3LB

(Received 20 August 1985)

Several points of interpretation are reviewed bearing on the celebrated discovery by Darwin (1953) that the added mass for a body translating uniformly in an infinite expanse of perfect fluid equals the drift-volume times the density of the fluid. The discussion focuses on the delicate qualifications needed to secure this equality as a mathematical proposition. In §2 a different approach to the matter is presented, leading to a new fact about added mass. In §3 a model of infinity in the fluid is proposed which clarifies an aspect of Darwin's original analysis.

1. Preamble

The added mass (or 'hydrodynamic mass') of a body moving in an infinite expanse of ideal fluid with constant density ρ is a tensor relating the fluid's kinetic energy K to the body's linear and angular velocities. For a rigid body, not necessarily axisymmetric, that moves in the x_1-direction without rotating, only the first diagonal component of this tensor is relevant and it is defined as $m_1 = 2K/c^2 = I_1/c$, where c is the velocity of the body. The quantity I_1 thus related to K is the x_1-component of impulse, being expressible by

$$I_1 = -\rho \int_S \Phi n_1 \,\mathrm{d}s, \tag{1}$$

where Φ is the evaluation of the velocity potential ϕ at the surface S of the body and n_1 is the x_1-component of the unit normal directed into the fluid (Kochin, Kibel & Roze 1964, p. 397). As is well known, I_1 cannot be identified in general with the total momentum M_1 of the fluid; but there are artifices, to be recalled presently, whereby the identity $I_1 = M_1$ may be rationalized.

Independently of such considerations, the only immediate kinematic attribute of I_1 is the following equation which follows from Green's theorem referred to the harmonic function ϕ. In terms of

$$C_1 = \int_\Delta x_1 \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3,$$

with Δ denoting the space inside S occupied by the body, the equation is

$$\frac{I_1}{\rho} + \frac{\mathrm{d}C_1}{\mathrm{d}t} + 4\pi A_1 = 0, \tag{2}$$

where A_1 is the coefficient of the dipole far field (i.e. in spherical polar coordinates $\phi \sim A_1 r^{-2} \cos\theta$ as $r \to \infty$). An interpretation explained by Benjamin (1986, §2) is that (2) reflects the Galilean invariance of the hydrodynamic problem. By itself, however, (2) is not very informative. A more interesting interpretation concerns drift as studied by Darwin (1953), about which primarily this note is written.

Considering the uniform motion of a body in the x_1-direction, Darwin defined the *drift-volume* D by reference to a surface composed of fluid particles that initially, when the body is far to the left, is a plane perpendicular to the x_1-axis. After passage of the body far to the right, this surface is displaced and together with the initial plane encloses the volume D. The main fact established in Darwin's paper is that, for an unbounded fluid, D is the same as m_1/ρ. This famous result is not in dispute. As a mathematical proposition, however, it depends crucially on arbitrary assumptions about the ordering of infinity in the fluid which, although nicely appreciated in Darwin's original treatment, have been obscured by some subsequent commentaries and which consequently deserve re-emphasis.

After opening with an incisively treated example, Darwin's account proceeded in steps to the general proof, first in two and then three space dimensions, and inherent ambiguities associated with conditionally convergent integrals were duly exposed. Specifically, he showed first that $\rho c D$ equals the volume integral expressing M_1 (1953, equations (4.8) and (8.8)); and then he showed that M_1 becomes determinate, equalling I_1 as given by (1), if and only if in the integral x_1 is taken to the infinite limits *before* the cross-sectional coordinates x_2 and x_3 are. In other words, to justify $M_1 = I_1$, the expanse of fluid has to be reckoned as a circular cylinder of length L and radius R with its generators in the x_1-direction, and with $L \rightarrow \infty$ and $R \rightarrow \infty$ *in this order*. Thus Darwin rediscovered a curious fact that had first been pointed out by Theodorsen (1941; see also Birkhoff 1950, chapter 5, §5).

If the limit $R \rightarrow \infty$ is taken before $L \rightarrow \infty$, or if the fluid is assigned a rigid outer boundary, however remote, the conclusion is that $D = -V$, where V is the volume of the body. Needless to say, moreover, values for D intermediate between m_1/ρ and $-V$ are obtainable by tailoring the conditions at infinity. This evasiveness of a comprehensive definition was reconciled in a wholly satisfactory way with Darwin's main result (1953, §7). He noted that most of the drift occurs near the path of the body, so that the equation $D = m_1/\rho$ can be approximated with arbitrary closeness by the contribution to D within a finite distance of the path. When the fluid is bounded rigidly far from the path, mass conservation requires that the total drift-volume is $-V$; but the necessary reflux $-V-(m_1/\rho)$ is spread over a very wide area.† It nevertheless remains as a cardinal fact of the subject that drift in an infinite fluid is not determinate without artificial specifications about infinity, just as total momentum is not. So, as an infinite fluid is a mathematical abstraction, secure deductions about its properties must depend on mathematical precision. The common-sense interpretation of Darwin's result $D = m_1/\rho$ is plain enough; but any proof of it in general is an inherently more exacting matter.

This note is prompted in part by a recent reappraisal of Darwin's result by Yih (1985), who presented two alternative demonstrations of it. To the present author, however, it seems that neither of these demonstrations takes proper account of the qualifications needed for a general theorem, as were made clear in Darwin's account. Both demonstrations depend on the evaluation of conditionally convergent integrals by choosing a particular order of integration or a particular shape of the boundary of the fluid at infinity. Although such particular choices may appear natural in some sense, they are difficult to justify rigorously.

The following section expounds a quite different approach to Darwin's result and leads to a new interpretation of added mass. An alternative rationale for removing the indeterminacy of integrals such as that for M_1 is also noted in §3.

† Not, however, just at the edges, as suggested by Darwin (1953, p. 350). Rather it is spread uniformly over the cross-section (see §3 below).

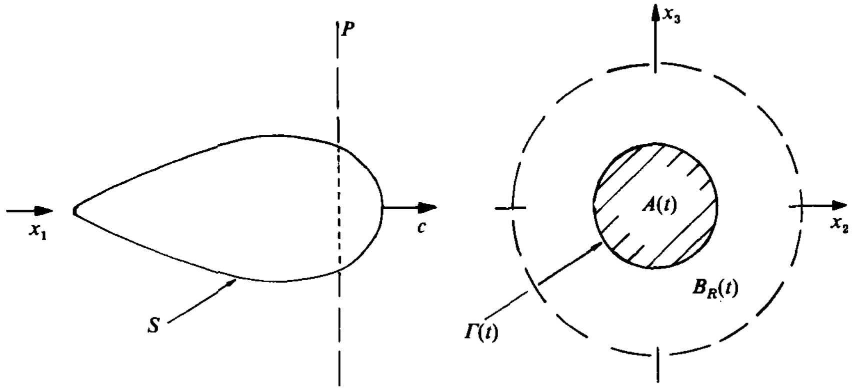

FIGURE 1. Illustration of symbols used in describing passage of solid body through plane P perpendicular to its path.

2. Drift reconsidered

The exemplary situation already specified is illustrated in figure 1. As indicated in the figure, let P denote any fixed plane perpendicular to x_1, and $A(t)$ denote the time-dependent point set $P \cap \Delta$, namely the cross-section of the moving body in the plane P. Also let $\Gamma(t) = P \cap S$ denote the time-dependent boundary of $A(t)$, typically a single closed curve in P as Δ passes through but at times possibly several such curves. Finally, let $B_R(t)$ denote a *bounded* open subset of $P \backslash A(t)$ such that the inner boundary of $B_R(t)$ is $\Gamma(t)$, its outer boundary is independent of t, and R is the radius of the largest circle contained in $B_R(\pm\infty)$. Thus, when $t \to -\infty$ and $A(t)$ is empty, B_R is a finite hole in P through which the body will ultimately pass during some phase of its motion, during which phase $B_R(t) = B_R(-\infty)\backslash A(t)$ becomes an annulus surrounding the cross-section $A(t)$ of the body.

Now, the total displacement of fluid volume through B_R, say D_R,† obviously accumulates at a varying rate equal to the volume flux through $B_R(t)$. So, writing $D_R(t)$ for the accumulation to time t, implying $D_R(t) \to 0$ as $t \to -\infty$ and $D_R(t) \to D_R$ as $t \to \infty$, we have

$$\frac{\mathrm{d}D_r(t)}{\mathrm{d}t} = \int_{B_R(t)} \frac{\partial\phi}{\partial x_1}\,\mathrm{d}x_2\,\mathrm{d}x_3. \tag{3}$$

In this integral $\partial\phi/\partial x_1$ is replaceable by $-c^{-1}\partial\phi/\partial t$, since $\phi = \phi(x_1 - ct, x_2, x_3)$ everywhere in the fluid for all t. To reduce the integral further, use can be made of the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R(t)} \phi\,\mathrm{d}x_2\,\mathrm{d}x_3 = \int_{B_R(t)} \frac{\partial\phi}{\partial t}\,\mathrm{d}x_2\,\mathrm{d}x_3 - \oint_{\Gamma(t)} \Phi\left(\frac{\partial\bar{x}_2}{\partial t}\,\mathrm{d}\bar{x}_3 - \frac{\partial\bar{x}_3}{\partial t}\,\mathrm{d}\bar{x}_2\right), \tag{4}$$

where in the contour integral Φ is the evaluation of ϕ on $\Gamma(t) \subset S$ and $\bar{x}_2$, $\bar{x}_3$ are the coordinates of points in $\Gamma(t)$ according to any parametric representation. In the light of the fact that the body's surface S is translating uniformly this contour integral is seen to equal another total derivative with respect to t, namely

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{S}(t)} \Phi n_1\,\mathrm{d}s, \tag{5}$$

† It needs to be emphasized that D_R is not identical with the contribution, respecting the area $B_R(-\infty)$, to the total drift-volume D as defined by Darwin. But plainly the two are made arbitrarily close by taking R large enough.

where the surface $\mathscr{S}(t)$ is just the part of S to the right of P at time t. The integral of (5) over $(-\infty, \infty)$ evidently recovers I_1/ρ as given by (1).

Combining this result with (3) as reduced by (4) and recalling the definition $m_1 = I_1/c$, we conclude that

$$\begin{aligned} D_R &= \frac{m_1}{\rho} + \frac{1}{c}\left(\lim_{t\to-\infty} - \lim_{t\to\infty}\right)\int_{B_R(t)} \phi \,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &= \frac{m_1}{\rho} + \frac{1}{c}\left(\lim_{x_1\to\infty} - \lim_{x_1\to-\infty}\right)\int_{B_R} \phi \,\mathrm{d}x_2\,\mathrm{d}x_3. \end{aligned} \tag{6}$$

Here B_R is written for $B_R(\infty) = B_R(-\infty)$.

While calling for care the interpretation of (6) is simpler than that of Darwin's analytical result. On the assumption that B_R is bounded, both limits on the right of (6) are zero if ϕ has only dipole strength at infinity. If an arbitrary function of time alone is added to ϕ, having of course no dynamical significance, the two limits cancel. Thus the result for the volume of fluid displaced through any finite hole B_R in P is

$$D_R = \frac{m_1}{\rho}. \tag{7}$$

This result accords with Darwin's for the total drift-volume D, which can be identified with $\lim_{R\to\infty} D_R$. The remarkable implication of the present result, however, is that D_R is independent of the (finite) size or shape of B_R, remaining the same even if B_R is shrunk to the smallest hole in P through which the body passes. Thus the added mass can be interpreted as the total displacement of mass through such a hole, and the whole of the drift-volume is similarly accountable.

On the other hand, let the preceding argument be retraced from the premise that $\lim_{R\to\infty} B_R(t) = P\backslash A(t)$ is the cross-section through which drift is to be reckoned. The outcome is again (6) but with the limit $R\to\infty$ preceding $x_1 \to \pm\infty$ on the right side. One then obtains

$$\begin{aligned} D_\infty &= \frac{m_1}{\rho} + \frac{4\pi A_1}{c} \\ &= -V, \end{aligned} \tag{8}$$

where the second line follows from the first by virtue of (2) and the obvious fact that $\mathrm{d}C_1/\mathrm{d}t = cV$. As must be expected, the same final result is obtained when the fluid is taken to be bounded by a rigid cylindrical surface aligned with the x_1-axis.

Although the different results (7) and (8) are exactly as encountered in Darwin's analysis, the priority of (7) is perhaps more conspicuous in the present derivation. Reliance on the tenuous identity $M_1 = cm_1$ has been avoided.

3. Alternative assumptions about infinity

One artifice for making total momentum determinate has been mentioned, namely that originally proposed by Theodorsen and used by Darwin. A mild objection may be levelled against it on the grounds that it does not comply with a basic symmetry of the hydrodynamic problem. Properties such as the relation between added mass and drift are obviously invariant under rotations of the body together with its direction of motion, but the geometry of infinity must also be rotated *ad hoc* in order to accommodate Darwin's line of argument. Another objection, equally marginal, is that the artifice contradicts a helpful model adopted by many writers who have used the concepts of impulse and added mass decisively for other purposes (e.g. Saffman 1967). The model assigns the fluid a *rigid* spherical boundary of indefinitely great radius, so that rotational symmetry is incorporated. But on this basis the correct but misleading conclusion is that $D = M_1/\rho c = -V$, and correspondingly for straight paths of a body in any other direction. Note, however, that the result (7) is unaffected.

This model has been discussed by Benjamin & Olver (1982, §6.5), who noted an alternative which is in fact well suited to present purposes. To allow for a rigid outer boundary at $r = \lambda$, the dipole far field is modified to

$$\phi^* = A_1 \cos\theta\left(\frac{1}{r^2}+\frac{2r}{\lambda^3}\right), \tag{9}$$

which is a harmonic function satisfying $\partial\phi^*/\partial n = 0$ at the boundary. Thus a uniform velocity $2A_1/\lambda^3$ in the x_1-direction is added to all parts of the fluid.† When λ is made large enough the modification becomes insignificant at any given r; but it ensures that integral properties such as M_1 have definite limits as $\lambda \to \infty$. Green's theorem shows that the volume integral expressing M_1/ρ equals the surface integral of $-x_1\,\partial\phi^*/\partial n$ over S, which is the same as the integral of $-x_1\,\partial\phi/\partial n$ in the limit $\lambda\to\infty$ and equals $-\mathrm{d}C_1/\mathrm{d}t = -cV$. The complementary surface integral over the outer boundary is zero because $\partial\phi^*/\partial n = 0$ there. The impulsive reaction of this fictitious boundary is given by

$$\frac{I_1^\infty}{\rho} = -\int_{r=\lambda} \phi^* n_1\,\mathrm{d}s = -3\int_{r=\lambda} \phi n_1\,\mathrm{d}s = -4\pi A_1,$$

where $n_1 = \cos\theta$ is the x_1-component of the outward unit normal. Since Green's theorem also shows that $M_1 = I_1 - I_1^\infty$, the general identity (2) thus reconfirms that $M_1 = -\rho\,\mathrm{d}C_1/\mathrm{d}t$ in the present case.

The alternative model takes the fluid to be bounded by an ideally compliant, 'pressure-release' surface $r = \lambda$, so that the far field is modified to

$$\phi^{**} = A_1 \cos\theta\left(\frac{1}{r^2}-\frac{r}{\lambda^3}\right). \tag{10}$$

Thus $I_1^\infty = 0$ and therefore $M_1 = I_1$ in this case. A uniform velocity $-A_1/\lambda^3$ is added to all parts of the fluid; and correspondingly the flow rate across the outer boundary is

$$\int_{r=\lambda} x_1 \frac{\partial\phi^{**}}{\partial n}\,\mathrm{d}s = -4\pi A_1 = \frac{I_1}{\rho}+\frac{\mathrm{d}C_1}{\mathrm{d}t},$$

which Green's theorem retraces to the equality $M_1 = I_1$.

Because of its rotational symmetry and presumably greater intuitive appeal, being based on a physical rather than abstract mathematical characterization of conditions at infinity, this second model has evident advantages as a support for Darwin's original line of argument – and also for Yih's. The model is just a conceptual nicety, however, not a radical reinforcement of the proposition $D = m_1/\rho$.

† Note that A_1 is negative for a body moving in the positive x_1-direction.

REFERENCES

BENJAMIN, T. B. 1986 Hamiltonian theory for motions of bubbles in an infinite liquid. *J. Fluid Mech.* (submitted).

BENJAMIN, T. B. & OLVER, P. J. 1982 Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.* **125**, 137–185.

BIRKHOFF, G. 1950 *Hydrodynamics*. Princeton University Press. (Dover edition 1955.)

DARWIN, C. 1953 Note on hydrodynamics. *Proc. Camb. Phil. Soc.* **49**, 342–354.

KOCHIN, N. E., KIBEL, I. A. & ROZE, N. V. 1964 *Theoretical Hydrodynamics*. Interscience.

SAFFMAN, P. G. 1967 The self-propulsion of a deformable body in a perfect fluid. *J. Fluid Mech.* **28**, 385–389.

THEODORSEN, T. 1941 Impulse and momentum in an infinite fluid. *Von Kármán anniversary volume*, p. 49. California Institute of Technology.

YIH, C.-S. 1985 New derivations of Darwin's theorem. *J. Fluid Mech.* **152**, 163–172.